Understanding Taylor and Maclaurin Series

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November 2024

1 Taylor Series

let $f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + ... + c_nx^n$

If I differentiate f(x) 3 times:

$$f'''(x) = 0 + 0 + 0 + 3 \cdot 2 \cdot 1 \cdot c_3 + 4 \cdot 3 \cdot 2c_4x + 5 \cdot 4 \cdot 3c_5x^2 + ...$$

We see that the terms before c_3 are differentiated to 0 because if you differentiate a constant you get 0 so if you differentiate x^2 3 times you get 0. The term with c_3 is a constant because differentiating x^3 3 times yields a constant. The terms after c_3 have an x in them, so when you plug in $x = 0$, they disappear:

$$f'''(x)|_{x=0} = 0 + 0 + 0 + 3 \cdot 2 \cdot 1 \cdot c_3 + \cancelto{0}{4 \cdot 3 \cdot 2c_4x} + \cancelto{0}{5 \cdot 4 \cdot 3c_5x^2} + ...$$

This leaves behind $f'''(x) = 3 \cdot 2 \cdot 1 \cdot c_3$.

In other words, if we differentiate f n times and evaluate at x=0:

$$f^{(n)}(x)|_{x=0} = \frac{d^n}{dx^n}(\underbrace{c_0 + c_1x + c_2x^2 + c_3x^3 + ...}_{\text{Becomes 0 from differentiation}} + \underbrace{c_nx^n}_{\text{Becomes constant}} + \underbrace{c_{n+1}x^{n+1} + ...}_{\text{Becomes 0 from plugging in x=0}})$$

The terms before c_n go to 0 because differentiating $x^{(\text{a number less than n})}$ n times yields 0. The terms after c_n go to 0 because differentiating $x^{(\text{a number greater than n})}$ n times yields a term with x which when plugging in x=0 yields 0. The only non-zero term is the term with c_n in it.

$$f^{(n)}(x)|_{x=0} = \frac{d^n}{dx^n}(c_nx^n)$$

Doing the power rule n times to x^n yields $n(n-1)(n-2)...(2)(1)x^0$, which is $n!$ so

$$f^{(n)}(x)|_{x=0} = c_nn!$$

This means the constant term c_n can be written as

$$c_n = \frac{1}{n!} f^{(n)}(x)|_{x=0}$$

We write $f^{(n)}(x)|_{x=0}$ as $f^{(n)}(0)$, so it cleans up a little:

$$c_n = \frac{1}{n!} f^{(n)}(0)$$

Remember with this notation that we are first differentiating f n times with respect to x and then plugging in $x = 0$.

Going to our original definition of $f(x)$, we can plug in for the c terms:

$$f(x) = \frac{1}{0!} f^{(0)}(0) + \frac{1}{1!} f^{(1)}(0)x + \frac{1}{2!} f^{(2)}(0)x^2 + \frac{1}{3!} f^{(3)}(0)x^3 + ... + \frac{1}{n!} f^{(n)}(0)x^n$$

$$= f(0) + f'(0)x + \frac{1}{2!} f''(0)x^2 + \frac{1}{3!} f'''(0)x^3 + ... + \frac{1}{n!} f^{(n)}(0)x^n$$

Using $f(x) = \sum_{n=0}^{\infty} c_n x^n$, we get $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)x^n$

2 Maclaurin Series

For the Taylor series, we centered f at 0. For the Maclaurin series, we can center f around any number a and say

$$f(a + h) = c_0 + c_1 h + c_2 h^2 + c_3 h^3 + ... + c_n h^n$$

f tells you what value you get as you travel variable distance h away from constant a. Just remember than we are differentiating f with respect to h in this case. Using the same logic as for the Taylor series, we get

$$f'''(a + h)|_{h=0} = 0 + 0 + 0 + 3 \cdot 2 \cdot 1 c_3 + \cancelto{0}{4 \cdot 3 \cdot 2h} + ...$$

$$c_3 = \frac{1}{3!} f'''(a + h)|_{h=0}$$

$$c_n = \frac{1}{n!} f^{(n)}(a + h)|_{h=0}$$

which can be written as

$$c_n = \frac{1}{n!} f^{(n)}(a)$$

This yields

$$f(a + h) = f(a) + f'(a)h + \frac{1}{2!} f''(a)h^2 + \frac{1}{3!} f'''(a)h^3 + ... + \frac{1}{n!} f^{(n)}(a)h^n$$

If we write $x = a + h$, we get

$$h = x - a$$

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3 + ... + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

Note about notation: $\frac{df}{dx} = \frac{df}{dh}$ because x is just h but offset but a constant a, so with $f^{(n)}(x)$ such as $f'''(x)$, differentiating with respect to either x or h is equivalent.